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COMMENT

Remark upon a random walk renormalisation

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Abstract. A direct renormalisation method, previously applied to the self-avoiding random walk problem, is shown to give exact results for the one-dimensional random walk problem for every choice of the cell size.

The self-avoiding random walk problem has recently been treated by a non-linear renormalisation method (Napiórkowski *et al* 1979). The method was based upon a direct comparison between walks on the original lattice and walks on a lattice with larger lattice constant. Figure 1 shows an example on the triangular lattice, with three original sites forming one site on the coarser lattice.

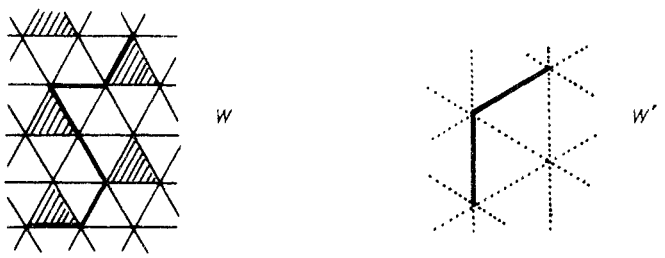

Figure 1. A walk on two levels for three-spin cells on the triangular lattice.

The same method can be applied to unrestricted random walks. This problem is simple, and there is no need to invoke renormalisation theory. In particular, the index ν, defined through the average square distance covered by N-step walks, $\langle R_N^2 \rangle^{1/2} \sim N^{\nu}$, is $\frac{1}{2}$. However, it may be interesting to demonstrate how the method works in a simple exactly solvable case. Thus motivated, we consider the one-dimensional random walk problem.

We collect groups of l consecutive sites on a linear lattice, called cells, into renormalised sites on the transformed lattice, and associate a unique random walk W' with each original walk W by declaring a renormalised site to be visited during W' if and only if a majority of its l sites is visited during W. (When precisely one-half of the sites are visited, one requires in addition that an arbitrarily selected site must be contained in W in order to have the visit registered. We take the selected site to be the site number $(\frac{1}{2}l + 1)$ in the cell.)

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The major simplification in the one-dimensional case is that a nearest-neighbour site walk W generates a nearest-neighbour cell walk W'. Thus the renormalisation transformation consists merely in determining the renormalised nearest-neighbour weight K' through

$$K' = \sum_{W} K^{n(W)} \tag{1}$$

where $n(W)$ is the number of steps in W and the sum runs over all the site walks compatible with a nearest-neighbour cell step. If this renormalisation transformation has a fixed point K^* with a relevant eigenvalue λ, the exponent ν is given by

$$\nu = \ln l / \ln \lambda. \tag{2}$$

In order to evaluate the RHS of (1) one has to consider random walks that start at a certain site, called the left-crossed site, and terminate at the corresponding site in the neighbouring cell, called the right-crossed site, without visiting corresponding selected sites in other cells. The right-crossed site (say) can be visited only once while the rest of the allowed sites can be visited an arbitrary number of times. The number of sites that can be visited an arbitrary number of times is $2l-1$.

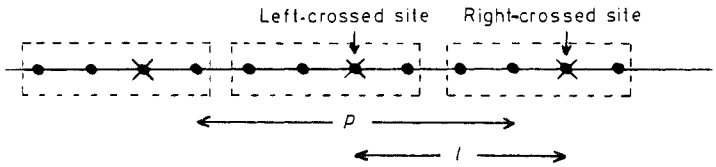

Figure 2. An example of partitions into cells of $l = 4$. Here $p = 7$.

To evaluate (1) we introduce two functions $f^n_{p,l}$ and $g^n_{p,l}$. The $g^n_{p,l}$ is the number of different n-step walks that end up l units to the right of the starting point such that the right endpoint together with p sites to the left of it are all available during the walks. The $f^n_{p,l}$ is defined as $g^n_{p,l}$ with the additional restriction that the right-end site is visited only once (i.e. at the final step). Defining the corresponding generating functions

$$F_{p,l}(K) = \sum_{n=0}^{\infty} f^n_{p,l} K^n \tag{3}$$

$$G_{p,l}(K) = \sum_{n=0}^{\infty} g^n_{p,l} K^n \tag{4}$$

the sum in equation (1) becomes

$$K' = F_{2l-1,l}(K). \tag{5}$$

Whenever $l = 0$ we find it convenient to drop this subscript in equations (3) and (4). Likewise we define $g^0_p = 1$ and $f^0_p = 0$. From the above definitions one finds that g^n_p and f^n_p are related by

$$g^n_p = 1 + f^n_p + \sum_{n_1} f^{n_1}_p f^{n-n_1}_p + \sum_{n_1}\sum_{n_2} f^{n_2}_p f^{n_1-n_2}_p f^{n-n_1}_p + \ldots \tag{6a}$$

which, by use of the generating functions (3) and (4), gives

$$G_p = \frac{1}{1-F_p}. \tag{6b}$$

$$G_{p,l} = \frac{F_{p,l}}{1-F_p} \qquad l = 1, 2, \ldots. \tag{6c}$$

Furthermore, one realises that

$$\begin{aligned} f^{n+2}_{p+1} &= g^n_p \\ f^{n+1}_{p+1,l+1} &= g^n_{p,l} \end{aligned} \tag{6d}$$

or

$$\begin{aligned} F_{p+1} &= K^2 G_p \\ F_{p+1,l+1} &= K G_{p,l}. \end{aligned} \tag{6e}$$

Knowing $F_1 = K^2$ one can thus generate all the $F_{p,l}$ and $G_{p,l}$ by use of these equations. Using (6) we get

$$\begin{aligned} F_{p+2,l+1} = K\frac{F_{p+1,l}}{1-F_{p+1}} &= K^{l+1}\frac{G_{p-l+1}}{\prod_{q=1}^{l}(1-F_{p-q+2})} \\ &= K^{l+1}\prod_{q=1}^{l+1} G_{p-q+2} = KF_{p,l}\frac{G_{p+1}G_p}{G_{p-l}} \end{aligned} \tag{7a}$$

and for $p = l = 1$ we obtain (defining $G_0 = 1$)

$$F_{3,2} = \frac{K^2}{1-2K^2}. \tag{7b}$$

Combining (6*b*) and (6*e*) we get

$$G_{p+1} = \frac{1}{1-K^2 G_p}. \tag{8}$$

Formulae (7) and (8) can be used as recursion relations to calculate $F_{p,l}$.

Using (7*b*) one checks that for $l = 2$ the renormalisation transformation (5) has a fixed point $K^* = \frac{1}{2}$ and the relevant eigenvalue $\lambda = l^2 = 2^2$. We shall now prove that this is a general property of the renormalisation transformation, i.e. for any integer l ($l > 1$) the transformation (5) has a fixed point $K^* = \frac{1}{2}$ and an eigenvalue $\lambda = l^2$. In our proof we shall use the following three facts:

$$G^*_p = G_p(K^*) = 2\frac{p+1}{p+2} \tag{9}$$

$$\frac{G^*_{p+1}G^*_p}{G^*_{p-l}} = 2 \qquad \text{for } p = 2l-1 \tag{10}$$

$$G'^*_p = \left.\frac{\mathrm{d}G_p(K)}{\mathrm{d}K}\right|_{K=K^*} = \frac{8}{3}\frac{p(p+1)}{p+2}. \tag{11}$$

It is easy to check that the above formulae hold for $p = 1$ and then using (8) one proves by induction that they hold for any p. Using formulae (7a) and (10) one checks for $p = 2l - 1$ that if

$$F^*_{p,l} = F_{p,l}(K^*) = \tfrac{1}{2} \tag{12}$$

then also

$$F^*_{p+2,l+1} = \tfrac{1}{2}. \tag{13}$$

Since property (12) holds for $p = 3$ and $l = 2$ we have thus shown that the renormalisation group equation (5) has a fixed point $K^* = \frac{1}{2}$. Now we shall show that if $F'^*_{p,l} = l^2$ then $F'^*_{p+2,l+1} = (l+1)^2$ for $p = 2l - 1$. Taking the derivative of both sides of equation (7a) and using equations (10)–(13) we obtain for $p = 2l - 1$:

$$\begin{aligned} F'^*_{p+2,l+1} &= F^*_{p+2,l+1}\left(\frac{1}{K^*} + \frac{F'^*_{p,l}}{F^*_{p,l}} + \frac{G'^*_{p+1}}{G^*_{p+1}} + \frac{G'^*_p}{G^*_p} - \frac{G'^*_{p-l}}{G^*_{p-l}}\right) \\ &= 1 + l^2 + \tfrac{2}{3}[(p+1) + p - (p-l)] = (l+1)^2. \end{aligned} \tag{14}$$

Thus we have shown by induction that the renormalisation group transformation for the one-dimensional random walk with an arbitrary choice of cell size has a fixed point $K^* = \frac{1}{2}$ with an eigenvalue equal to the square of the number of sites in one cell. This leads to the result that for a random walk ν is equal to $\frac{1}{2}$, the well-known exact result.

Renormalisation of two-dimensional unrestricted random walks should yield similar exact results: $\nu = \frac{1}{2}$ and K_c being equal to the inverse coordination number. To show this seems to be a much more complicated task and remains a challenge.

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